Math 152-AN Hour Exam #4 Satake

Name: KEY

Read all directions and problems *carefully*! Show all appropriate work for credit.

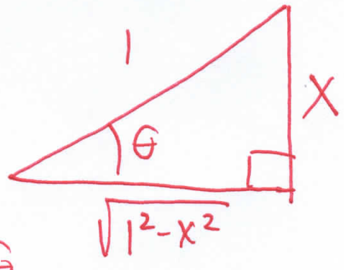
1. Evaluate the following integral using Trig Substitutions. (21 pts.)

$$\int \frac{\sqrt{1^2-x^2}}{x^4}\,dx$$

$\sin\theta = \frac{x}{1}$ $\quad x=\sin\theta$

$dx = \cos\theta\, d\theta$

$$\int \frac{\cos\theta}{(\sin\theta)^4}\cdot\frac{\cos\theta}{1}\,d\theta$$



$\cos\theta = \frac{\sqrt{1^2-x^2}}{1}$

$$\int \frac{\cos^2\theta}{\sin^2\theta}\cdot\frac{1}{\sin^2\theta}\,d\theta$$

$u=\cot\theta$

$$\int \cot^2\theta\cdot\csc^2\theta\, d\theta \qquad du = -\csc^2\theta\, d\theta$$

$$= -\int u^2\,du = -\frac{u^3}{3}+C = -\frac{\cot^3\theta}{3}+C = \boxed{-\frac{1}{3}\left(\frac{\sqrt{1-x^2}}{x}\right)^3+C}$$

+21

2. Five the partial fraction decomposition for the function. **Do not** integrate. (12 pts.)

$$\frac{5x^2+20x+6}{x^3+2x^2+x} = \frac{5x^2+20x+6}{x(x^2+2x+1)} = \frac{5x^2+20x+6}{x(x+1)^2}$$

$$= \frac{A}{x}+\frac{B}{x+1}+\frac{C}{(x+1)^2} = \boxed{\frac{6}{x}+\frac{-1}{x+1}+\frac{9}{(x+1)^2}}$$

+12

$$5x^2+20x+6 = A(x+1)^2+Bx(x+1)+Cx$$
$$= Ax^2+2Ax+A+Bx^2+Bx+Cx$$

$$\begin{cases} A+B=5 \\ 2A+B+C=20 \\ A=6 \end{cases} \Rightarrow B=-1, \quad C=9$$

33

3. Use the method of partial fractions to evaluate the following integral. (18 pts.)

$\int \frac{x^2-1}{x^3+x}dx \Rightarrow \frac{X^2-1}{X(X^2+1)} = \frac{A}{X} + \frac{BX+C}{X^2+1} \Rightarrow$ $A+B=1$

$C=0 \Rightarrow B=2$

$X^2-1 = AX^2+A + BX^2+CX$ $A=-1$

$= \int \frac{-1}{X}dx + \int \frac{2X}{X^2+1}dx$ $u=x^2+1$

$du=2xdx$

$= -\ln|X| + \int \frac{1}{u}du = -\ln|X| + \ln|u| + C$

$= \boxed{-\ln|X| + \ln|X^2+1| + C} \overset{OR}{=} \ln\left|\frac{X^2+1}{X}\right| + C$ (+18)

4. Evaluate the following Improper Integrals or state that they diverge. (22 pts.)

$\int_{-\infty}^{0} \frac{2}{x^2+1}dx$

$2 \cdot \lim_{a \to -\infty} \int_a^0 \frac{1}{x^2+1^2}dx$

$2 \cdot \lim_{a \to -\infty} \tan^{-1}x \Big|_a^0$

$2 \lim_{a \to -\infty} \left[\tan^{-1}0 - \tan^{-1}a\right]$

HORIZONTAL ASYMPTOTE

$= 2\left[0 - \left(-\frac{\pi}{2}\right)\right]$

$= \boxed{\pi u^2}$ (+11)

$\int_0^1 \frac{1}{\sqrt[3]{x}}dx$

$\lim_{a \to 0^+} \int_a^1 x^{-\frac{1}{3}}dx$

$\lim_{a \to 0^+} \left[\frac{3}{2}x^{\frac{2}{3}}\right]_a^1$

$= \frac{3}{2}\lim_{a \to 0^+}\left[1^{\frac{2}{3}} - a^{\frac{2}{3}}\right]$

$= \frac{3}{2}\left[1-0\right]$

$= \boxed{\frac{3}{2}u^2}$ (+11)

40

5. Find the specific solution of the following Initial Value problem. (15 pts.)
$y'(t) = -2y + 3$, $y(0) = 1$

$y(t) = Ce^{-2t} - \frac{3}{-2}$

$y(t) = Ce^{-2t} + \frac{3}{2}$

$y(0) = Ce^{-2(0)} + \frac{3}{2} = 1$

$C = 1 - \frac{3}{2}$

$C = -\frac{1}{2}$

$\boxed{y(t) = -\frac{1}{2}e^{-2t} + \frac{3}{2}}$

+15

6. Find the general solution of the separable differential equation $\frac{dy}{dx} = \frac{e^{2x}}{y}$. Verify that your solution is in fact a solution of the differential equation. (12 pts.)

$y\,dy = e^{2x}dx$

$\int y\,dy = \int e^{2x}dx$

$2\left(\frac{y^2}{2} + C_1 = \frac{e^{2x}}{2} + C_2\right)2$

$y^2 + C_1 = e^{2x} + C_2$

$y^2 = e^{2x} + C$

$\boxed{y = \pm\sqrt{e^{2x} + C}}$

VERIFY

$\pm\frac{1}{\cancel{2}}(e^{2x}+C)^{-\frac{1}{2}} \cdot e^{2x}(\cancel{2}) = \frac{e^{2x}}{\pm\sqrt{e^{2x}+C}}$

$\pm\frac{e^{2x}}{\sqrt{e^{2x}+C}} = \pm\frac{e^{2x}}{\sqrt{e^{2x}+C}}$ ✓

+12

Solution of a First-Order Linear Differential Equation

For $y'(t) = ky + b$, the solution is $y = Ce^{kt} - \frac{b}{k}$

27